Matrices are extremely useful tools that allow engineers to succinctly state and solve a variety of problems. Some of these areas include:

- Circuits and Circuit Design
- Imaging and Coding
- Signal Processing
- Optics
- Electromagnetism
- Differential Equations and Linear Systems

We have already been introduced to matrices in previous chapters in a limited manner. Now we explore matrices in more detail.

An \((m \times n)\) matrix is a rectangular table of elements (numbers). The rectangular table has \(m\) rows horizontal) and \(n\) columns (vertical) as below.

\[
\begin{bmatrix}
A_{11} & A_{12} & A_{13} & \cdots & A_{1n} \\
A_{21} & A_{22} & A_{23} & \cdots & A_{2n} \\
A_{31} & A_{32} & A_{33} & \cdots & A_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A_{m1} & A_{m2} & A_{m3} & \cdots & A_{mn}
\end{bmatrix}
\]

Any element of this matrix can be determined by specifying its row and column address. For example, \(A_{33}\) is the intersection of the third row and third column.

We note that a column vector is a special case of a matrix \((n \times 1)\) and a row vector is a \((1 \times n)\) matrix. Also, if \(m=n\), the matrix is called a square matrix. As we will see, square matrices have a special place in the theory of matrices so we will explore them in more detail.

Since, we have already discussed how to input matrices and how to manipulate them; we will start by looking at their properties and applications.

**Adding and Subtracting Matrices**

To add or subtract matrices, they must have the same dimension.

\[
\begin{bmatrix}
2 & 3; 1 & 6
\end{bmatrix}
\]
A =
\[
\begin{pmatrix}
2 & 3 \\
1 & 6
\end{pmatrix}
\]
» B=[1  -2;  4  6]
B =
\[
\begin{pmatrix}
1 & -2 \\
4 & 6
\end{pmatrix}
\]
» C=A+B
C =
\[
\begin{pmatrix}
3 & 1 \\
5 & 12
\end{pmatrix}
\]
» D=A-B
D =
\[
\begin{pmatrix}
1 & 5 \\
-3 & 0
\end{pmatrix}
\]

Notice that this can be written as \( C_{ij} = A_{ij} + B_{ij} \), \( D_{ij} = A_{ij} - B_{ij} \).

**Question:** What happens if the sizes are different?

**Matrix Multiplication**

We have seen one type of matrix multiplication defined as array multiplication. This can be written as \( C_{ij} = A_{ij}B_{ij} \)

C=A.*B
C =
\[
\begin{pmatrix}
2 & -6 \\
4 & 36
\end{pmatrix}
\]

Another matrix multiplication can be defined in the following
Consider two matrices A (m x n) and B (n x p)
These matrices can be decomposed as follows.

\[
\tilde{A} = \begin{bmatrix}
\tilde{A}_1 \\
\tilde{A}_2 \\
\vdots \\
\tilde{A}_m
\end{bmatrix}, \quad \tilde{B} = [\tilde{B}_1 \ \tilde{B}_2 \ \tilde{B}_3 \ \ldots \ \tilde{B}_p]
\]

**How long are the A vectors, the B vectors?**
The matrix product C=AB is then defined as follows. The (i,j) element of C is given as the inner product (see Vectors) of the i\(^{\text{th}}\) row of A with the j\(^{\text{th}}\) column of B. \( \tilde{C}_{ij} = \tilde{A}_i \cdot \tilde{B}_j \) \( 1 \leq i \leq m, 1 \leq j \leq p \). Since these vectors must have the same length to be defined, that is why the number of
columns of A must equal the number of rows of B. With this definition, the resultant matrix has size (m x p).

**Exercises:**
What is the size of the resultant matrix if
A=(2 x 4) B=(4 x 3)
A=(2 x 2) B=(2 x 3)
A=(4 x 2) B=(3 x 2)

**Example**
Consider the matrix multiplication of the (2x3) by (3x2). The resulting matrix is a (2x2) matrix.
To find the (1,2) element, we take the inner product of the first row of A with the second column of B (Look at the Blue Colored elements) This equals \((2)(3)+(3)(-2)+(1)(8)=8\). Therefore, C(1,2)=8.

\[
\begin{array}{ccc}
2 & 3 & 1 \\
4 & -2 & 3 \\
\end{array}
\]

\[
\begin{array}{cc}
4 & 3 \\
7 & -2 \\
7 & 8 \\
\end{array}
\]

Which element does this combination give?

\[
\begin{array}{ccc}
2 & 3 & 1 \\
4 & -2 & 3 \\
\end{array}
\]

\[
\begin{array}{cc}
4 & 3 \\
7 & -2 \\
7 & 8 \\
\end{array}
\]
Do all 4 elements of C and compare to the result below which is done in Matlab.

```matlab
» A=[2 3 1;4 -2 3]
A =
    2   3   1
    4  -2   3

» B=[4 3;7 -2; 7 8]
B =
    4   3
    7  -2
    7   8

» C=A*B % Look at the multiplication symbol (*). This is different than A.*B where the multiplication symbol is given as (.*).
C =
    36   8
    23  40
```

Matrix multiplication has some interesting properties.

1. The follow the distributive law $A(B+C)=AB+AC$;
2. Matrices do not in general commute. $AB \neq BA$

**Problems**

1. Find 2 (2 x 2) matrices which commute;
2. Find a matrix $M$ such that $AM=A$ for any (2 x 2) matrix $A$;
3. Verify the associative law
4. A diagonal matrix has non-zero elements only along the main diagonal. Determine the product of any two diagonal matrices.

**Matrix Transposition**

The transpose of a matrix $B = A^T$ is defined as $B_{ji} = A_{ij}$ so that the rows and columns are interchanged. The transposition operator is the same as that for vectors (').

```matlab
A =
    3   1
    2   5
    1   6

» B=A'
B =
    3   2   1
    1   5   6
```
A matrix is symmetric if the matrix is the same as the transpose. i.e. \( A = A^T \)

\[
A = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}
\]

A symmetric matrix must be square. Why?

**Problems**
Verify the following properties of transposition.

1. \( (A^T)^T = A \)
2. \( (A+B)^T = A^T + B^T \)
3. \( (AB)^T = B^T A^T \)
4. \( B = A + A^T \) is a symmetric matrix.

**Matrix Inversion**

Matrix Division does not really exist. However if we think of \( C = A/B \) as \( A \cdot B^{-1} \), then matrix division can be defined if the inverse of a matrix is defined. This operation is called Right Division. Similarly, we can define \( C = A \backslash B \) as \( (A^{-1}) \cdot B \) or Left Division.

When we talk about a multiplicative inverse for scalars, what we are looking for is a number \( A^{-1} \) such that \( A^{-1}A = 1 \).

Why 1? What makes the number 1 so special?
The number 1 is special because \( A \cdot 1 = A \) for any \( A \). Therefore, 1 is called the identity element of the real numbers. Therefore, the inverse of a number is the number such that \( \text{inverse} \cdot \text{number} = \text{identity} \).

What is the matrix equivalent of Identity?
The Identity matrix is a matrix such that \( A \cdot I = A \) for any \( A \). The I symbol is used to denote the identity.

**Problems**

1. Verify that \[
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\] is an identity matrix for all 2 x 2 square matrices
2. Explain why any identity matrix must be square.
3. Verify that \[
\begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}
\] is not an identity matrix for all \(2 \times 2\) square matrices.

4. What is the form of the identity for \(N \times N\) matrix?

Now the matrix inverse \(A^{-1}\) is defined as \(A^{-1}A=AA^{-1}=I\).

Suppose I try to find the matrix inverse by direct multiplication

**Example:** Find the inverse of \[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\]

Solution

Since \[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\begin{bmatrix}
a' & b' \\
c' & d'
\end{bmatrix}=
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\]

We get 4 equations in the 4 primed unknowns.
The (1,1) element equation is \(aa'+bc'=1\) with similar equations for the other elements.
This should convince you that finding a matrix inverse is **hard** to do.

Matlab has a built in function to find the inverse. The **inv** function.

**Example:**

Find the inverse of \[
\begin{bmatrix}
4 & -2 & 6 \\
3 & 1 & 5 \\
-1 & 2 & 8
\end{bmatrix}
\]

\(a = \)

\[
\begin{bmatrix}
4 & -2 & 6 \\
3 & 1 & 5 \\
-1 & 2 & 8
\end{bmatrix}
\]

```matlab
>> b=inv(a)
b =
\begin{bmatrix}
-0.0217 & 0.3043 & -0.1739 \\
-0.3152 & 0.4130 & -0.0217 \\
0.0761 & -0.0652 & 0.1087
\end{bmatrix}
```

```matlab
>> b*a  % This must be the identity!!!!
ans =
\begin{bmatrix}
1.0000 & 0 & -0.0000 \\
-0.0000 & 1.0000 & -0.0000 \\
-0.0000 & 0 & 1.0000
\end{bmatrix}
```
This is the identity matrix which verifies that the inverse is correct.

**Problems:**

1. What is the inverse of a diagonal matrix?
2. Prove that if a matrix has a column or row whose elements are all zeros, the inverse does not exist. This is in analogy to saying that the number “0” does not have an inverse.
3. Prove that if any column is a multiple of another column or a row is a multiple of another row, the matrix does not have an inverse. For example \[
\begin{bmatrix}
1 & 3 \\
3 & 9
\end{bmatrix}
\] does not have an inverse.

**Using Matrices to Solve Simultaneous Linear Equations**

Consider the system of equations

\[
\begin{align*}
x + y &= 6 \\
x - 2y &= 3
\end{align*}
\]

The solution to this is simple and can be obtained graphically or by back-substitution.

The answer is \(x=5, y=1\).

**Problem:**

Show that the system of equations can be written as

\[
\begin{bmatrix}
1 & 1 \\
1 & -2
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix} =
\begin{bmatrix}
6 \\
3
\end{bmatrix}
\]

or symbolically as

\(\bar{A} \times \bar{x} = \bar{b}\). Here \(A\) is a 2 x 2 matrix, \(x\) is (2 x 1) vector of unknowns and \(b\) is the (2 x 1) vector whose elements are the RHS of the equations.

If we symbolically operate on this matrix equation by \(A^{-1}\), we get \(\bar{x} = A^{-1}\bar{b}\).

To see that this does give us the solution we obtained before

\(b = \)
6
3

» a=[1 1; 1 -2]
a =
1 1
1 -2
» x=inv(a)*b
x =
5 %x=5
1 % y=1

Problem

Solve
X-3Y+4Z=2;
2X+Y-5Z=10;
X+Z=2 % be careful. Think of this as X + 0Y +Z=2

Singular Matrices

In the problem above, it should be clear that if A had no inverse, no solution is possible. Let us explore this a little more carefully. The general problem (2x2) problem

\[ ax + by = m \]
\[ cx + dy = n \]

can be thought of geometrically as finding the intersection of the two lines described by the two simultaneous equations. In general, the lines on a plane intersect at only one unique point giving us the solution. However, if the two lines have the same slope, it is possible to have zero points of intersection or an infinite number (the lines overlap).

Problem:

1. Determine the condition in which there is no unique point of intersection.
2. How are these conditions related to the conditions that the coefficient matrix is singular (has no inverse).
The same geometrical way of viewing a system of linear equations can be extended to any number of variables. For simplicity, consider the case of 3 equations in 3 variables. The geometrical way of viewing this problem is that the solution is described by the simultaneous intersection of 3 planes in a 3-Dimensional space. Again, in most cases, a unique point of intersection will occur. However, we can imagine that there are cases in which this will not be so. For example, all 3 planes are parallel. In another case, suppose that the first two planes intersect. The points of intersection form a line. Now, the third plane may cross the line at a point or can miss the line altogether. If the plane misses the line, no solution can occur. It is clear that the conditions describing the non-intersection is more difficult than the 2-dimensional case.

**Determinants**

It is clearly important to be able to determine if a coefficient matrix is singular. If it is, we are wasting our time finding a solution to the Matrix Equation $Ax = b$. If the matrix is not singular, a solution must exist. It would be very helpful if we could determine by inspection whether the matrix is singular. In previous homework exercises, we showed that a condition of singularity could be determined. For the general 2 x 2 case, we showed that a matrix is singular if a certain function of the elements $F(a, b, c, d) = (ad - bc) = 0$. If the order of the matrix is greater, we could find with greater effort, a function of the matrix elements that when set to zero is the condition of singularity. This method is very cumbersome. Instead, we look for a numerical approach to determine whether a matrix is singular. We use the MATLAB det function. This function numerically evaluates the coefficient functions discussed in the previous paragraph. Therefore, if det(A) is zero, the matrix is singular. Otherwise, the matrix is non-singular and a unique solution exists. Consider, first the non-singular case.

```matlab
>> A=[1 2 6;4 5 3; 7 8 9]
>> det (A)
    ans = -27
```

Consider now a singular case
Problems:
1. Show by numerical examples, that the determinant of a 2x2 matrix is \((ad - bc)\).
2. Show numerically, the following properties of determinants
   a) \(\text{Det}(AB) = \text{Det}(A) - \text{Det}(B)\).
   b) \(\text{Det}(I) = 1\).
   c) From parts a) and b) show that \(\text{Det}(A^{-1}) = \frac{1}{\text{Det}(A)}\)
3. Discuss the Matrix equation \(Ax = 0\) when the matrix \(A\) is
   a) Non-singular b) Singular

An Application to Circuit Analysis
In Engr. 204, you will study circuits carefully and develop ways to solve circuits made up of various resistor combinations and sources (Batteries). It turns out that the set of unknowns (in this case, the node voltages of the circuit) satisfies a set of linear equations and therefore, we can use the matrix methods to solve for them.
You can use the method described in the previous section to find the values of voltages at each node of a circuit and then the current in each branch.
Consider the circuit diagram below.

![Circuit Diagram](image)

Circuit #1
Without going into details, it can be shown that the node voltages satisfy the following set of equations.

\[V_1 = 5\]
\[-6V_1 + 10V_2 - 3V_3 = 0\]
\[-V_2 + 51V_3 = 0\]
Note that $V_1$ is just the battery voltage that just happens to be 5 in this case.

Finally, once all the node voltages are known, all currents can be found using Kirchoff’s Law that states that the current $I = \frac{\Delta V}{R}$ where $\Delta V = V_{tail} - V_{head}$ where tail and head signify the current direction chosen.

**Example**

The lamp current $i_{lamp} = i_2 = \frac{V_3 - 0}{2} = \frac{V_3}{2}$. If the current arrow is drawn in the opposite direction, the current is negative $i_{lamp} = i_2 = \frac{0 - V_3}{2} = -\frac{V_3}{2}$.

$V_3$ is found by solving the system of equations.

As another example, consider the circuit below which has two sources

Circuit #2
The equations for this circuit are

$V_1 = 7$
$-2V_1 + 7V_2 - 4V_3 = 0$
$-6V_2 + 10V_3 = 10$
$V_3 - V_4 = 10$
An Application to Circuit Design

Many times, instead of being given all the inputs, we need to choose the inputs so that a desired output is obtained. As an example, suppose that in circuit 1, we want to determine the value of the resistance R (before we fixed it to 100 ohms) such that the current in the lamp is between 0.05 and 0.075 Amperes. To solve this design problem, we must first construct the system function that relates the desired output (the lamp current) to the unknown control parameter (the variable resistance R).

This is done by writing a function that accepts R as input and returns as output the current ilamp.

This function is called circuit and it is saved in a file called circuit.m which is constructed as follows.

```matlab
function ilamp=circuit(R)
    A=[1 0 0;(-6*R) (7*R+300) -300;0 -2 (R+2)]
    % Matrix storing node equations
    B=[5;0;0]; % Vector holding right side quantities
    V=inv(A)*B; % Solving for unknown voltages
    V3=V(3); % Isolating the voltage across the lamp
    ilamp=V3/2; % Calculating current in lamp
```

In MATLAB, we can now create an array that holds all the admissible resistor values R, solve for the current array Ilampv and plot the results. By inspecting the resulting curve, we can determine the admissible resistance values to maintain the lamp current within the specified boundaries.

```matlab
>> for n=1:100
    resv(n)=n; % each resistor value is stored
    ilampv(n)=circuit(resv(n)); % current for each resistor value
end
>> plot(resv,ilampv) % Plot current versus resistance
```

Note that the value of R must be greater than zero to protect the lamp so our initial set of admissible resistances are positive and greater than zero. Resistance must be positive on physical grounds. Your figure should show that the admissible resistance values are 15<R<40 ohms. Why could you not perform the resistance loop using array operations?