Lecture 17


**Objectives:**
1. Concepts of the direct and diffuse (scattered) solar radiation.
2. Source function and a radiative transfer equation for the diffuse solar radiation.
4. Legendre polynomial expansion of the scattering phase function.

**Required reading:**
L02: 3.4, 6.1, Appendix E

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1. **Concepts of the direct and diffuse solar radiation.**
   - The solar radiation field is traditionally considered as a sum of two distinctly different components: **direct** and **diffuse**: \( I = I_{\text{dir}} + I_{\text{dif}} \)

**Direct solar radiation** is a part of solar radiation filed that has survived the extinction passing a layer with optical depth \( \tau_* \) and it obeys the Beer-Bouguer-Lambert (extinction) law:

\[
I_{\text{dir}} = I_0 \exp(-\tau_* / \mu_0) \tag{17.1}
\]

where \( I_0 \) is the solar intensity at a given wavelength at the top of the atmosphere and \( \mu_0 \) is a cosine of the solar zenith angle \( \theta_0 \) (\( \mu_0 = \cos(\theta_0) \)).

The **direct solar flux** is

\[
F_{\text{dir}} = \mu_0 F_0 \exp(-\tau_* / \mu_0) \tag{17.2}
\]
2. Source function and a radiative transfer equation for the diffuse solar radiation.

Diffuse radiation arises from the light that undergoes one scattering event (single scattering) or many (multiple scattering).

Recall Lecture 3 where we defined the source function

\[ J_\lambda = (\dot{j}_{\lambda, \text{thermal}} + \dot{j}_{\lambda, \text{scattering}}) / \beta_{e, \lambda} \]

where \( \dot{j}_{\lambda, \text{thermal}} \) is the thermal emission (\( j_{\lambda, \text{thermal}} = \beta_{a, \lambda} B_\lambda(T) \)) and \( \dot{j}_{\lambda, \text{scattering}} \) is the re-radiation from multiple scattering.

Using the volume scattering coefficient \( \beta_{s, \lambda} \) and the phase function \( P(\mu, \varphi, \mu', \varphi') \), we have

\[ j_{\lambda, \text{scattering}}(\Omega) = \frac{\beta_{s, \lambda}}{4\pi} \int \Omega' P(\Omega', \Omega') d\Omega' \]  \[ \text{[17.3]} \]

NOTE: Recall the scattering phase function \( P(\mu, \varphi, \mu', \varphi') \) (i.e., the element of the scattering matrix \( P_{11} \)) represents the angular distribution of scattered energy as a function of direction. By the definition (see Lecture 13), it is normalized as

\[ \frac{1}{4\pi} \int_{\Omega} P(\cos \Theta) d\Omega = 1 \]

where \( \Theta \) is the scattering angle

\[ \cos(\Theta) = \cos(\Theta')\cos(\Theta) + \sin(\Theta')\sin(\Theta) \cos(\varphi' - \varphi) = \mu \mu' + (1-\mu^2)\frac{1}{2}(1-\mu'^2)^{1/2} \cos(\varphi' - \varphi) \]
Thus the **source function for diffuse solar radiation** may be written as two components

\[
J(\tau, \mu, \varphi) = \frac{\omega_0}{4\pi} \int_{0}^{2\pi} \int_{-1}^{1} I(\tau, \mu', \varphi') P(\mu, \varphi, \mu', \varphi') d\mu' d\varphi' + \frac{\omega_0}{4\pi} F_0 P(\mu, \varphi, -\mu_0, \varphi_0) \exp\left(-\frac{\tau}{\mu_0}\right)
\]

[17.4]

where the \(\omega_0\) is the single scattering albedo and \(P\) is the scattering phase function.

**NOTE:** In Eq.[17.4], the first term on the right-hand side shows that the phase function redirects the incoming intensity in the direction \((\mu', \varphi')\) to the direction \((\mu, \varphi)\), and the integrals account for all possible scattering events within the \(4\pi\) solid angle.

- The **source function for scattering** Eq.[17.4] is more complicated than a thermal source function:
  
  (i) It involves conditions throughout the atmosphere, while the thermal source function depends on local conditions only;

  (ii) The phase function \(P(\mu, \varphi, \mu', \varphi')\) may be a very complex function of the directions (and, in general, state of polarization).

Recall the radiative transfer equation defined in Lecture 3 for a plane-parallel atmosphere

\[
\mu \frac{dI_\lambda(\tau; \mu, \varphi)}{d\tau} = I_\lambda(\tau; \mu, \varphi) - J_\lambda(\tau; \mu, \varphi)
\]

Thus, using the source function for scattering, we can write the **radiative transfer equation for the diffuse radiation** as (omitting the subscript \(\text{dif}\) in \(I\))

\[
\mu \frac{dI(\tau, \Omega)}{d\tau} = I(\tau, \Omega) - \frac{\omega_0}{4\pi} \int_{4\pi} I(\tau, \Omega') P(\Omega, \Omega') d\Omega' - \frac{\omega_0}{4\pi} F_0 P(\Omega, -\Omega_0) \exp\left(-\frac{\tau}{\mu_0}\right)
\]

[17.5]
NOTE: Eq.[17.5] is an integro-differential equation. To solve Eq.[17.5], one needs to know the scattering coefficient $\beta_{s,\lambda}$, absorption coefficient $\beta_{a,\lambda}$, and scattering phase function $P(\mu, \varphi, \mu', \varphi')$ as a function of wavelength in each atmospheric layer.

Eq.[17.5] can be simplified if there is no dependency on the azimuth angle. For azimuthally independent case, we may define the phase function as

$$P(\mu, \mu') = \frac{1}{2\pi} \int_0^{2\pi} P(\mu, \varphi, \mu', \varphi') d\varphi'$$  \hspace{1cm} [17.6]

Using Eq.[17.6], we can write the azimuthally independent radiative transfer equation for the diffuse radiation

$$\frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{\alpha_0}{2} \int_{-1}^{1} I(\tau, \mu') P(\mu, \mu') d\mu' - \frac{\alpha_0}{4\pi} F_0 P(-\mu_0) \exp(-\tau / \mu_0)$$  \hspace{1cm} [17.7]

➢ To find a solution of the radiative transfer equation for diffuse radiation (i.e., to solve Eq.[17.5] or [17.7]), various approximate and “exact” techniques have been developed:

**Approximate methods:**

i) Single scattering approximations (this lecture)

ii) Two-stream approximations (Lecture 18)

iii) Eddington and Delta-Eddington approximations (Lecture 18)

**“Exact” methods:**

i) Discrete-ordinate technique (Lecture 20)

ii) Adding-doubling technique (Lecture 21)

iii) Monte-Carlo technique (Lecture 22)

If light has been scattered only once, the source function from Eq.[17.3] becomes

\[ J(\tau, \mu, \varphi) = \frac{\omega_0}{4\pi} F_0 P(\mu, \varphi, -\mu_0, \varphi_0) \exp(-\tau / \mu_0) \]  \[ [17.8] \]

and using the solution (derived in Lecture 3) of the radiation transfer in a plane-parallel atmosphere bounded by on two sides at \( \tau = 0 \) and \( \tau = \tau^* \):

for upward intensity (reflected)

\[ I_{\uparrow} (\tau; \mu; \varphi) = I_{\uparrow} (\tau^*; \mu; \varphi) \exp(-\frac{\tau^* - \tau}{\mu}) + \frac{1}{\mu} \int_{\tau}^{\tau^*} \exp(-\frac{\tau' - \tau}{\mu}) J_{\uparrow} (\tau'; \mu; \varphi) d\tau' \]

and downward intensity (transmitted)

\[ I_{\downarrow} (\tau; -\mu; \varphi) = I_{\downarrow} (0; -\mu; \varphi) \exp(-\frac{\tau}{\mu}) + \frac{1}{\mu} \int_{0}^{\tau} \exp(-\frac{\tau - \tau'}{\mu}) J_{\downarrow} (\tau'; -\mu; \varphi) d\tau' \]

we can write the solution for diffuse radiation in a single scattering approximation as

\[ I_{\uparrow} (\tau; \mu; \varphi) = I_{\uparrow} (\tau^*; \mu; \varphi) \exp(-\frac{\tau^* - \tau}{\mu}) + \frac{1}{\mu} \frac{\omega_0}{4\pi} F_0 P(\mu, \varphi, -\mu_0, \varphi_0) \int_{\tau}^{\tau^*} \exp(-[\frac{\tau' - \tau}{\mu} + \frac{\tau'}{\mu_0}]) d\tau' \]  \[ [17.9a] \]

\[ I_{\downarrow} (\tau; -\mu; \varphi) = I_{\downarrow} (0; -\mu; \varphi) \exp(-\frac{\tau}{\mu}) + \frac{1}{\mu} \frac{\omega_0}{4\pi} F_0 P(-\mu, \varphi, -\mu_0, \varphi_0) \int_{0}^{\tau} \exp(-[\frac{\tau' - \tau}{\mu} + \frac{\tau'}{\mu_0}]) d\tau' \]  \[ [17.9b] \]

Assuming that there is no diffuse downward radiation at the top of the atmosphere

\[ I_{\downarrow} (0; -\mu, \varphi) = 0 \]

and no upward diffuse radiation at the surface (i.e., no reflection from the surface)

\[ I_{\uparrow} (\tau^*; \mu, \varphi) = 0 \]  \[ [17.10] \]
Then from Eq.[17.9a,b] for a finite atmosphere with optical depth $\tau_*$, we find the **reflected and transmitted diffuse intensities**

$$I_{\downarrow}^{\uparrow}(0, \mu, \varphi) = \frac{\omega_0 \mu_0 F_0}{4 \pi (\mu + \mu_0)} P(\mu, \varphi, -\mu_0, \varphi_0) \left[ 1 - \exp \left( -\tau_*(\frac{1}{\mu} + \frac{1}{\mu_0}) \right) \right]$$ \[17.11\]

and for $\mu$ is NOT equaled to $\mu_0$

$$I_{\downarrow}^{\uparrow}(\tau^*, -\mu, \varphi) = \frac{\omega_0 \mu_0 F_0}{4 \pi (\mu - \mu_0)} P(-\mu, \varphi, -\mu_0, \varphi_0) \left[ \exp \left( -\frac{\tau_*}{\mu} \right) - \exp \left( -\frac{\tau_*}{\mu_0} \right) \right]$$ \[17.12\]

and for $\mu = \mu_0$

$$I_{\downarrow}^{\uparrow}(\tau^*, -\mu, \varphi) = \frac{\omega_0 \mu_0 F_0}{4 \pi \mu_0} P(-\mu_0, \varphi_0, -\mu_0, \varphi_0) \left[ \exp \left( -\frac{\tau_*}{\mu_0} \right) \right]$$ \[17.13\]

- For the single scattering approximation, the diffuse intensities are directly proportional to the phase function.

**NOTE**: The single scattering approximation is valid for the optically thin atmosphere (i.e., small optical depth).

For $\tau^* \ll 1$ (called the single scattering approximation in remote sensing), Eq.[17.11] simplifies to

$$I_{\downarrow}^{\uparrow}(0, \mu, \varphi) = \frac{\omega_0}{4 \pi} F_0 P(\Theta) \frac{\tau_*}{\mu}$$

4. **Legendre polynomial expansion of the scattering phase function.**

For many practical applications, the phase function must be numerically expanded in Legendre polynomials with a finite number of terms $N$ as

$$P(\cos \Theta) = \sum_{l=0}^{N} \sigma_l^* P_l(\cos \Theta)$$ \[17.14\]

where $\Theta$ is the scattering angle

$$\cos(\Theta) = \cos(\Theta') \cos(\Theta) + \sin(\Theta') \sin(\Theta) \cos(\Phi' - \Phi) = \mu' \mu + (1 - \mu'^2)^{1/2}(1 - \mu^2)^{1/2} \cos(\Phi' - \Phi)$$

and $\sigma_l^*$ is the expansion coefficients expressed as

$$\sigma_l^* = \frac{2l + 1}{2} \int_{-1}^{1} P(\cos \Theta) P_l(\cos \Theta) d \cos(\Theta), \ l=0, 1, \ldots, N$$ \[17.15\]
NOTE: Orthogonal properties of the Legendre polynomials:

\[
\int_{-1}^{1} P_l(\cos\Theta) P_k(\cos\Theta) d\cos(\Theta) = 0 \text{ for } l \neq k
\]

\[
\int_{-1}^{1} P_k(\cos\Theta) P_l(\cos\Theta) d\cos(\Theta) = \frac{2}{2l+1} \text{ for } l = k
\]

The first few Legendre polynomials are given by:

\[ P_0(x) = 1 \]
\[ P_1(x) = x \]
\[ P_2(x) = \frac{1}{2} (3x^2 - 1) \]
\[ P_3(x) = \frac{1}{2} (5x^3 - 3x) \]
\[ P_4(x) = \frac{1}{8} (35x^4 - 30x^2 + 3) \]
\[ P_5(x) = \frac{1}{8} (63x^5 - 70x^3 + 15x) \]
Rewriting the radiative transfer equation in terms of associated Legendre polynomials:

Eq.[17.14] can be expressed in the terms of associated Legendre polynomials

\[
P (\mu, \varphi, \mu', \varphi') = \sum_{m=0}^{N} \sum_{l=m}^{N} \sigma_{m}^{l} P_{l}^{m} (\mu) P_{l}^{m} (\mu') \cos (m (\varphi' - \varphi))
\]

where

\[
\sigma_{m}^{l} = (2 - \delta_{0,m}) \sigma_{l}^{*} \frac{(l-m)!}{(l+m)!}
\]

\[l=m, \ldots, N; \quad 0 \leq m \leq N\]

and \(\delta_{0,m}\) is the Kronecker delta: \(\delta_{0,m} = 1\) for \(m=0\) and otherwise \(\delta_{0,m} = 0\).

In similar manner, we may expand the diffuse intensity in the cosine series

\[
I (\tau, \mu, \varphi) = \sum_{m=0}^{N} I_{m}^{m} (\tau, \mu) \cos (m (\varphi_{0} - \varphi))
\]

Using Eqs.[17.16] and [17.17] and the orthogonality of the associated Legendre polynomials, the equation of the radiative transfer for the diffuse intensity (Eq.[17.7]) splits into \((N+1)\) independent equations in the form

\[
\mu \frac{dI^{m} (\tau, \mu)}{d\tau} = I^{m} (\tau, \mu) - (1 + \delta_{0,m}) \frac{\omega_{0}}{4} \int_{-1}^{1} P_{l}^{m} (\mu) I^{m} (\tau, \mu') d\mu' -\]

\[- \frac{\omega_{0}}{4\pi} \sum_{i=m}^{N} \sigma_{i}^{m} P_{i}^{m} (\mu) P_{i}^{m} (-\mu) F_{0} \exp (-\tau / \mu_{0})
\]

\[
m=0 \Rightarrow \text{azimuthal independent case:}
\]

From Eq.[17.16], the azimuth-independent phase function can be expressed as

\[
P (\mu, \mu') = \sum_{l=0}^{N} \sigma_{l} P_{l} (\mu) P_{l} (\mu')
\]

For this case Eq.[17.18] simplifies to (omitting the superscript 0 for \(m=0\))

\[
\mu \frac{dI (\tau, \mu)}{d\tau} = I (\tau, \mu) - \frac{\omega_{0}}{2} \sum_{i=0}^{N} \sigma_{i}^{*} P_{i} (\mu) \int_{-1}^{1} P_{i} (\mu') I (\tau, \mu') d\mu' -\
\]

\[- \frac{\omega_{0}}{4\pi} \sum_{l=0}^{N} \sigma_{l}^{*} P_{l} (\mu) P_{l} (-\mu) F_{0} \exp (-\tau / \mu_{0})
\]
Expansion of the Henyey-Greenstein phase function in the Legendre polynomials

The Henyey-Greenstein scattering phase function is a model phase function, which is often used in radiative transfer calculations:

\[
P_{HG} (\cos \Theta) = \frac{1 - g^2}{(1 + g^2 - 2g \cos \Theta)^{3/2}}
\]

\(g\) is the asymmetry parameter.

Let’s take \(g=0.5\) (representative of aerosols)

Legendre expansion:

\[
P_{HG} (\cos \Theta) = \sum_{l=0}^{N} \varpi^*_l P_l (\cos \Theta)
\]

\[
\varpi^*_l = \frac{2l + 1}{2} \int_{-1}^{1} P_{HG} (\cos \Theta) P_l (\cos \Theta) d \cos(\Theta)
\]

If \(N=0\) \(\Rightarrow\)

\[
\varpi^*_0 = \frac{1}{2} \int_{-1}^{1} P_{HG} (\cos \Theta) P_0 (\cos \Theta) d \cos(\Theta) = \frac{1}{2} \int_{-1}^{1} P_{HG} (\cos \Theta) \ 1 \ d \cos(\Theta) = \frac{1}{2} \ 2 = 1
\]

NOTE: \(\varpi^*_0\) is always 1 because of normalization of the phase function

\[
\frac{1}{2} \int_{-1}^{1} P(\cos \Theta) d \cos(\Theta) = 1
\]

Thus, in the case of one term in the expansion, we have

\[
P_{HG} (\cos \Theta) \approx \sum_{l=0}^{0} \varpi^*_l P_l (\cos \Theta) = 1 * 1 = 1
\]

Plot (below) shows that using only one term gives poor approximation

NOTE: In all plots below, the black curve shows the Henyey-Greenstein phase function, and the red curve is for the Legendre expansion, all as a function of \(\cos(\Theta)\).
If N=1 =>

$$\omega_o^* = \frac{1}{2} \int_{-1}^{1} P_{HG}(\cos \theta) P_0(\cos \theta) d \cos(\theta) = \frac{1}{2} \int_{-1}^{1} P_{HG}(\cos \theta) \ 1 \ d \cos(\theta) = \frac{1}{2} \ 2 = 1$$

$$\omega_1^* = \frac{1}{2} \int_{-1}^{1} P_{HG}(\cos \theta) P_1(\cos \theta) d \cos(\theta) = \frac{1}{2} \int_{-1}^{1} P_{HG}(\cos \theta) (\cos \theta) \ d \cos(\theta) = 1.5$$

Thus, in the case of two terms in the expansion, we have

$$P_{HG}(\cos \theta) \approx \sum_{l=0}^{1} \omega_l^* P_l(\cos \theta) = 1 + 1.5(\cos \theta) \text{ - still accuracy is not so good!}$$
We can continue by including more terms to get a desirable accuracy
N=7-10 gives good approximation to the Henyey-Greenstein scattering phase function with g=0.5. The larger g, the larger number of terms will be required to achieve acceptable accuracy.