

## Lecture 17

### Principles of multiple scattering in the atmosphere. Radiative transfer equation with scattering for solar radiation in a plane-parallel atmosphere.

#### Objectives:

1. Concepts of the direct and diffuse (scattered) solar radiation.
2. Source function and a radiative transfer equation for the diffuse solar radiation.
3. Single scattering approximation.
4. Legendre polynomial expansion of the scattering phase function.

#### Required reading:

L02: 3.4, 6.1, Appendix E

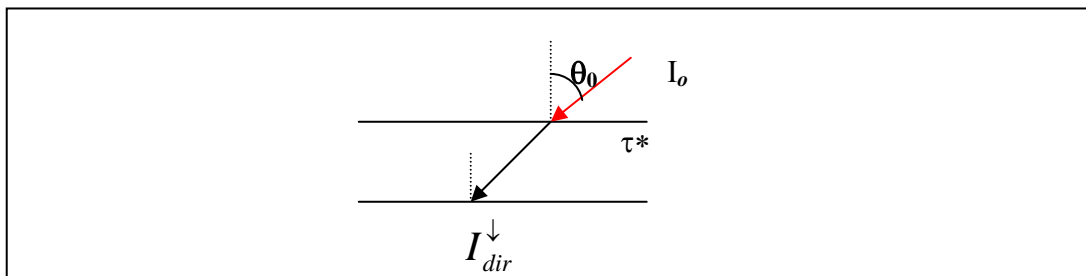
#### 1. Concepts of the direct and diffuse solar radiation.

- The solar radiation field is traditionally considered as a sum of two distinctly different components: **direct** and **diffuse**:  $I = I_{dir} + I_{dif}$

**Direct solar radiation** is a part of solar radiation field that has survived the extinction passing a layer with optical depth  $\tau^*$  and it obeys the Beer-Bouguer-Lambert (extinction) law:

$$I_{dir}^{\downarrow} = I_0 \exp(-\tau^* / \mu_0) \quad [17.1]$$

where  $I_0$  is the solar intensity at a given wavelength at the top of the atmosphere and  $\mu_0$  is a cosine of the solar zenith angle  $\theta_0$  ( $\mu_0 = \cos(\theta_0)$ ).

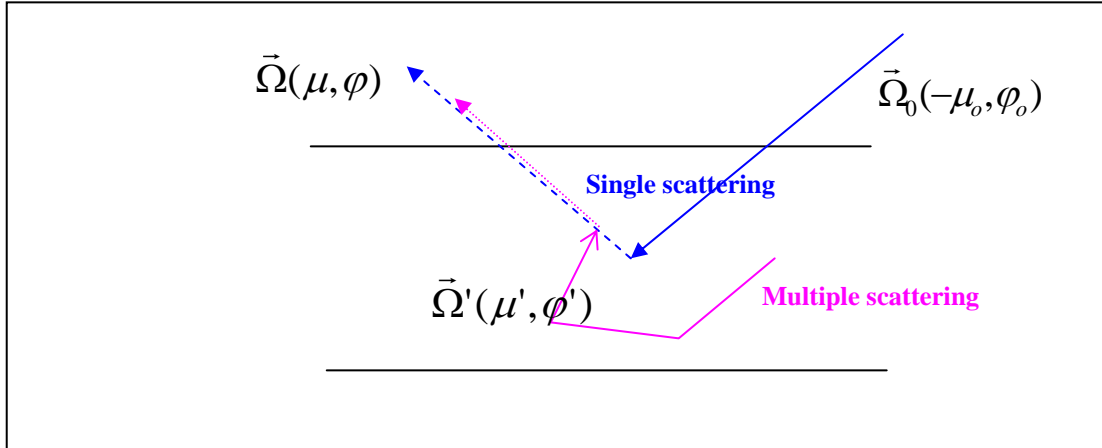


The **direct solar flux** is

$$F_{dir}^{\downarrow} = \mu_0 F_0 \exp(-\tau^* / \mu_0) \quad [17.2]$$

## 2. Source function and a radiative transfer equation for the diffuse solar radiation.

**Diffuse radiation** arises from the light that undergoes one scattering event (**single scattering**) or many (**multiple scattering**).



Recall Lecture 3 where we defined the source function

$$\mathbf{J}_\lambda = (\mathbf{j}_{\lambda, \text{thermal}} + \mathbf{j}_{\lambda, \text{scattering}}) / \beta_{e,\lambda}$$

where  $\mathbf{j}_{\lambda, \text{thermal}}$  is the **thermal emission** ( $j_{\lambda, \text{thermal}} = \beta_{a,\lambda} B_\lambda(T)$ )

and  $\mathbf{j}_{\lambda, \text{scattering}}$  is the re-radiation from multiple scattering.

Using the volume scattering coefficient  $\beta_{s,\lambda}$  and the phase function  $P(\mu, \varphi, \mu', \varphi')$ , we have

$$j_{\lambda, \text{scattering}}(\bar{\Omega}) = \frac{\beta_{s,\lambda}}{4\pi} \int_{\bar{\Omega}'} I(\bar{\Omega}') P(\bar{\Omega}, \bar{\Omega}') d\Omega' \quad [17.3]$$

**NOTE:** Recall the **scattering phase function**  $P(\mu, \varphi, \mu', \varphi')$  (i.e., the element of the scattering matrix  $P_{11}$ ) represents the angular distribution of scattered energy as a function of direction. By the definition (see Lecture13), it is normalized as

$$\frac{1}{4\pi} \int_{\Omega} P(\cos \Theta) d\Omega = 1$$

where  $\Theta$  is the scattering angle

$$\cos(\Theta) = \cos(\theta')\cos(\theta) + \sin(\theta')\sin(\theta) \cos(\varphi' - \varphi) = \mu'\mu + (1-\mu'^2)^{1/2}(1-\mu^2)^{1/2} \cos(\varphi' - \varphi)$$

Thus the **source function for diffuse solar radiation** may be written as two components

$$J(\tau, \mu, \varphi) = \frac{\omega_0}{4\pi} \int_0^1 \int_{-1}^1 I(\tau, \mu', \varphi') P(\mu, \varphi, \mu', \varphi') d\mu' d\varphi' + \frac{\omega_0}{4\pi} F_0 P(\mu, \varphi, -\mu_0, \varphi_0) \exp(-\tau / \mu_0) \quad [17.4]$$

where the  $\omega_0$  is the single scattering albedo and  $P$  is the scattering phase function.

**NOTE:** In Eq.[17.4], the first term on the right-hand side shows that the phase function redirects the incoming intensity in the direction  $(\mu', \varphi')$  to the direction  $(\mu, \varphi)$ , and the integrals account for all possible scattering events within the  $4\pi$  solid angle.

- The **source function for scattering** Eq.[17.4] is more complicated than a thermal source function:
  - (i) It involves conditions throughout the atmosphere, while the thermal source function depends on local conditions only;
  - (ii) The phase function  $P(\mu, \varphi, \mu', \varphi')$  may be a very complex function of the directions (and, in general, state of polarization).

Recall the radiative transfer equation defined in Lecture 3 for a plane-parallel atmosphere

$$\mu \frac{dI_\lambda(\tau; \mu, \varphi)}{d\tau} = I_\lambda(\tau; \mu, \varphi) - J_\lambda(\tau; \mu, \varphi)$$

Thus, using the source function for scattering, we can write the **radiative transfer equation for the diffuse radiation** as (omitting the subscript *dif* in  $I$ )

$$\mu \frac{dI(\tau, \bar{\Omega})}{d\tau} = I(\tau, \bar{\Omega}) - \frac{\omega_0}{4\pi} \int_{4\pi} I(\tau, \bar{\Omega}') P(\bar{\Omega}, \bar{\Omega}') d\Omega' - \frac{\omega_0}{4\pi} F_0 P(\bar{\Omega}, -\bar{\Omega}_0) \exp(-\tau / \mu_0) \quad [17.5]$$

**NOTE:** Eq.[17.5] is an integro-differential equation. To solve Eq.[17.5], one needs to know the scattering coefficient  $\beta_{s,\lambda}$ , absorption coefficient  $\beta_{a,\lambda}$  and scattering phase function  $P(\mu, \varphi, \mu', \varphi')$  as a function of wavelength in each atmospheric layer.

Eq.[17.5] can be simplified if there is no dependency on the azimuth angle.

For azimuthally independent case, we may define the phase function as

$$P(\mu, \mu') = \frac{1}{2\pi} \int_0^{2\pi} P(\mu, \varphi, \mu', \varphi') d\varphi' \quad [17.6]$$

Using Eq.[17.6], we can write **the azimuthally independent radiative transfer equation for the diffuse radiation**

$$\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - \frac{\omega_0}{2} \int_{-1}^1 I(\tau, \mu') P(\mu, \mu') d\mu' - \frac{\omega_0}{4\pi} F_0 P(\mu, -\mu_0) \exp(-\tau / \mu_0) \quad [17.7]$$

- **To find a solution of the radiative transfer equation for diffuse radiation** (i.e., to solve Eq.[17.5] or [17.7]), various approximate and “exact” techniques have been developed:

***Approximate methods:***

- i) Single scattering approximations (this lecture)
- ii) Two-stream approximations (Lecture 18)
- iii) Eddington and Delta-Eddington approximations (Lecture 18)

***“Exact” methods:***

- i) Discrete-ordinate technique (Lecture 20)
- ii) Adding-doubling technique (Lecture 21)
- iii) Monte-Carlo technique (Lecture 22)

### 3. Single scattering approximation.

If light has been scattered only once, the source function from Eq.[17.3] becomes

$$J(\tau, \mu, \varphi) = \frac{\omega_0}{4\pi} F_0 P(\mu, \varphi, -\mu_0, \varphi_0) \exp(-\tau / \mu_0) \quad [17.8]$$

and using the solution (derived in Lecture 3) of the radiation transfer in a plane-parallel atmosphere bounded by on two sides at  $\tau=0$  and  $\tau=\tau^*$ :

for upward intensity (reflected)

$$I_{\lambda}^{\uparrow}(\tau; \mu; \varphi) = I_{\lambda}^{\uparrow}(\tau^*; \mu; \varphi) \exp\left(-\frac{\tau^* - \tau}{\mu}\right) + \frac{1}{\mu} \int_{\tau}^{\tau^*} \exp\left(-\frac{\tau' - \tau}{\mu}\right) J_{\lambda}^{\uparrow}(\tau'; \mu; \varphi) d\tau'$$

and downward intensity (transmitted)

$$I_{\lambda}^{\downarrow}(\tau, -\mu, \varphi) = I_{\lambda}^{\downarrow}(0, -\mu, \varphi) \exp\left(-\frac{\tau}{\mu}\right) + \frac{1}{\mu} \int_0^{\tau} \exp\left(-\frac{\tau - \tau'}{\mu}\right) J_{\lambda}^{\downarrow}(\tau', -\mu, \varphi) d\tau'$$

we can write **the solution for diffuse radiation in a single scattering approximation** as

$$I_{\lambda}^{\uparrow}(\tau; \mu; \varphi) = I_{\lambda}^{\uparrow}(\tau^*, \mu, \varphi) \exp\left(-\frac{\tau^* - \tau}{\mu}\right) + \frac{1}{\mu} \frac{\omega_0}{4\pi} F_0 P(\mu, \varphi, -\mu_0, \varphi_0) \int_{\tau}^{\tau^*} \exp\left(-\left[\frac{\tau' - \tau}{\mu} + \frac{\tau'}{\mu_0}\right]\right) d\tau' \quad [17.9a]$$

$$I_{\lambda}^{\downarrow}(\tau; -\mu; \varphi) = I_{\lambda}^{\downarrow}(0, -\mu, \varphi) \exp\left(-\frac{\tau}{\mu}\right) + \frac{1}{\mu} \frac{\omega_0}{4\pi} F_0 P(-\mu, \varphi, -\mu_0, \varphi_0) \int_0^{\tau} \exp\left(-\left[\frac{\tau' - \tau}{\mu} + \frac{\tau'}{\mu_0}\right]\right) d\tau' \quad [17.9b]$$

Assuming that there is no diffuse downward radiation at the top of the atmosphere

$$I_{\lambda}^{\downarrow}(0, -\mu, \varphi) = 0$$

and no upward diffuse radiation at the surface (i.e., no reflection from the surface)

$$I_{\lambda}^{\uparrow}(\tau^*, \mu, \varphi) = 0 \quad [17.10]$$

Then from Eq.[17.9a,b] for a finite atmosphere with optical depth  $\tau^*$ , we find the **reflected and transmitted diffuse intensities**

$$I_{\lambda}^{\uparrow}(0, \mu, \varphi) = \frac{\omega_0 \mu_0 F_0}{4\pi(\mu + \mu_0)} P(\mu, \varphi, -\mu_0, \varphi_0) \left[ 1 - \exp\left(-\tau^* \left(\frac{1}{\mu} + \frac{1}{\mu_0}\right)\right) \right] \quad [17.11]$$

and for  $\mu$  is NOT equaled to  $\mu_0$

$$I_{\lambda}^{\downarrow}(\tau^*, -\mu, \varphi) = \frac{\omega_0 \mu_0 F_0}{4\pi(\mu - \mu_0)} P(-\mu, \varphi, -\mu_0, \varphi_0) \left[ \exp\left(-\frac{\tau^*}{\mu}\right) - \exp\left(-\frac{\tau^*}{\mu_0}\right) \right] \quad [17.12]$$

and for  $\mu = \mu_0$

$$I_{\lambda}^{\downarrow}(\tau^*, -\mu, \varphi) = \frac{\omega_0 \tau^* F_0}{4\pi\mu_0} P(-\mu_0, \varphi_0, -\mu_0, \varphi_0) \left[ \exp\left(-\frac{\tau^*}{\mu_0}\right) \right] \quad [17.13]$$

- For the single scattering approximation, the diffuse intensities are directly proportional to the phase function.

**NOTE:** The single scattering approximation is valid for the optically thin atmosphere (i.e., small optical depth).

For  $\tau^* \ll 1$  (called the single scattering approximation in remote sensing), Eq.[17.11]

simplifies to

$$I^{\uparrow}(0, \mu, \varphi) = \frac{\omega_0}{4\pi} F_0 P(\Theta) \frac{\tau^*}{\mu}$$

#### **4. Legendre polynomial expansion of the scattering phase function.**

For many practical applications, the phase function must be numerically expanded in Legendre polynomials with a finite number of terms  $N$  as

$$P(\cos \Theta) = \sum_{l=0}^N \varpi_l^* P_l(\cos \Theta) \quad [17.14]$$

where  $\Theta$  is the scattering angle

$$\cos(\Theta) = \cos(\theta')\cos(\theta) + \sin(\theta')\sin(\theta) \cos(\varphi' - \varphi) = \mu' \mu + (1 - \mu'^2)^{1/2} (1 - \mu^2)^{1/2} \cos(\varphi' - \varphi)$$

and  $\varpi_l^*$  is the expansion coefficients expressed as

$$\varpi_l^* = \frac{2l+1}{2} \int_{-1}^1 P(\cos \Theta) P_l(\cos \Theta) d \cos(\Theta), \quad l=0, 1, \dots, N \quad [17.15]$$

**NOTE:** Orthogonal properties of the Legendre polynomials:

$$\int_{-1}^1 P_k(\cos\Theta)P_l(\cos\Theta)d\cos(\Theta) = 0 \text{ for } l \neq k$$

$$\int_{-1}^1 P_k(\cos\Theta)P_l(\cos\Theta)d\cos(\Theta) = \frac{2}{2l+1} \text{ for } l = k$$

The first few Legendre polynomials are given by:

$$P_0(x) = 1$$

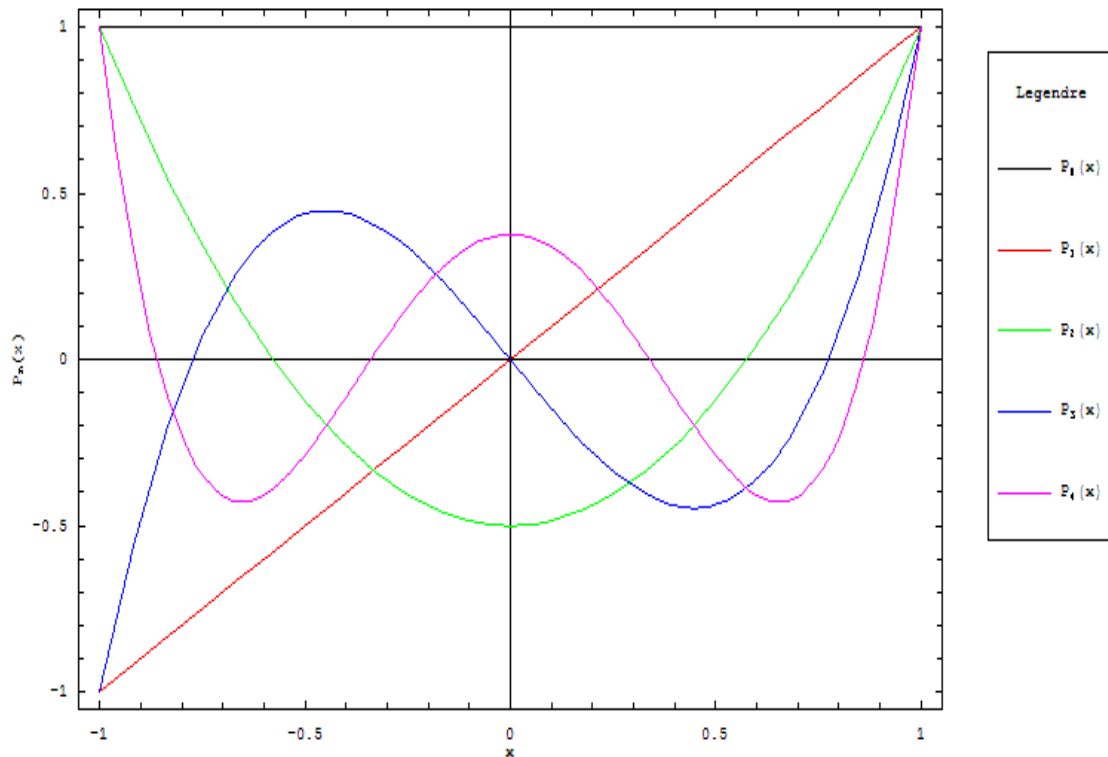
$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$



Rewriting the radiative transfer equation in terms of associated Legendre polynomials:

Eq.[17.14] can be expressed in the terms of associated Legendre polynomials

$$P(\mu, \varphi, \mu', \varphi') = \sum_{m=0}^N \sum_{l=m}^N \varpi_l^m P_l^m(\mu) P_l^m(\mu') \cos(m(\varphi' - \varphi)) \quad [17.16]$$

where

$$\varpi_l^m = (2 - \delta_{0,m}) \varpi_l^* \frac{(l-m)!}{(l+m)!} \quad l=m, \dots, N; \quad 0 \leq m \leq N$$

and  $\delta_{0,m}$  is the Kronecker delta:  $\delta_{0,m} = 1$  for  $m=0$  and otherwise  $\delta_{0,m} = 0$ .

In similar manner, we may expand the diffuse intensity in the cosine series

$$I(\tau, \mu, \varphi) = \sum_{m=0}^N I^m(\tau, \mu) \cos(m(\varphi_0 - \varphi)) \quad [17.17]$$

Using Eqs.[17.16] and [17.17] and the orthogonality of the associated Legendre polynomials, the equation of the radiative transfer for the diffuse intensity (Eq.[17.7]) splits into (N+1) independent equations in the form

$$\begin{aligned} \mu \frac{dI^m(\tau, \mu)}{d\tau} &= I^m(\tau, \mu) - (1 + \delta_{0,m}) \frac{\omega_0}{4} \sum_{l=m}^N \varpi_l^m P_l^m(\mu) \int_{-1}^1 P_l^m(\mu') I^m(\tau, \mu') d\mu' - \\ &- \frac{\omega_0}{4\pi} \sum_{l=m}^N \varpi_l^m P_l^m(\mu) P_l^m(-\mu_0) F_0 \exp(-\tau / \mu_0) \end{aligned} \quad [17.18]$$

**m=0 => azimuthal independent case:**

From Eq.[17.16], the azimuth-independent phase function can be expressed as

$$P(\mu, \mu') = \sum_{l=0}^N \varpi_l P_l(\mu) P_l(\mu') \quad [17.19]$$

For this case Eq.[17.18] simplifies to (omitting the superscript 0 for  $m=0$ )

$$\begin{aligned} \mu \frac{dI(\tau, \mu)}{d\tau} &= I(\tau, \mu) - \frac{\omega_0}{2} \sum_{l=0}^N \varpi_l^* P_l(\mu) \int_{-1}^1 P_l(\mu') I(\tau, \mu') d\mu' - \\ &- \frac{\omega_0}{4\pi} \sum_{l=0}^N \varpi_l^* P_l(\mu) P_l(-\mu_0) F_0 \exp(-\tau / \mu_0) \end{aligned}$$

[17.20]

## Expansion of the Henyey-Greenstein phase function in the Legendre polynomials

The Henyey-Greenstein scattering phase function is a model phase function, which is often used in radiative transfer calculations:

$$P_{HG}(\cos\Theta) = \frac{1 - g^2}{(1 + g^2 - 2g \cos\Theta)^{3/2}}$$

$g$  is the asymmetry parameter.

Let's take  $g=0.5$  (representative of aerosols)

Legendre expansion:

$$P_{HG}(\cos\Theta) = \sum_{l=0}^N \varpi_l^* P_l(\cos\Theta) \quad \text{and}$$

$$\varpi_l^* = \frac{2l+1}{2} \int_{-1}^1 P_{HG}(\cos\Theta) P_l(\cos\Theta) d \cos(\Theta)$$

**If N=0** =>

$$\varpi_0^* = \frac{1}{2} \int_{-1}^1 P_{HG}(\cos\Theta) P_0(\cos\Theta) d \cos(\Theta) = \frac{1}{2} \int_{-1}^1 P_{HG}(\cos\Theta) 1 d \cos(\Theta) = \frac{1}{2} \cdot 2 = 1$$

NOTE:  $\varpi_0^*$  is always 1 because of normalization of the phase function

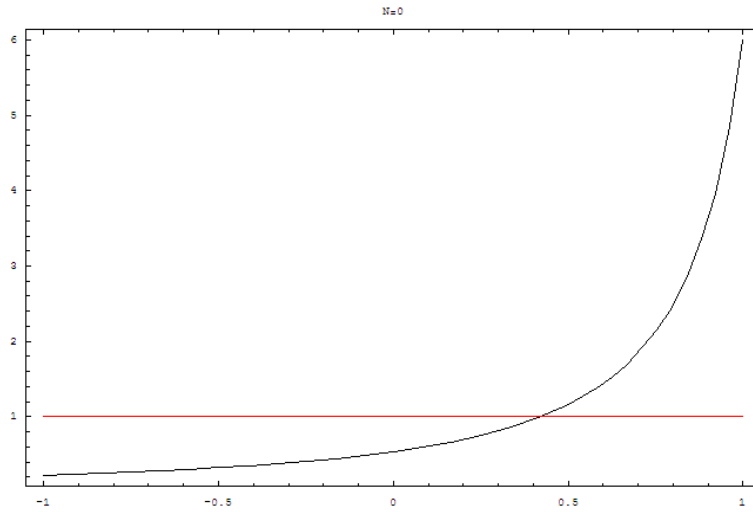
$$\frac{1}{2} \int_{-1}^1 P(\cos\Theta) d \cos(\Theta) = 1$$

Thus, in the case of one term in the expansion, we have

$$P_{HG}(\cos\Theta) \approx \sum_{l=0}^0 \varpi_l^* P_l(\cos\Theta) = 1 * 1 = 1$$

Plot (below) shows that using only one term gives poor approximation

**NOTE:** In all plots below, the black curve shows the Henyey-Greenstein phase function, and the red curve is for the Legendre expansion, all as a function of  $\cos(\Theta)$ .



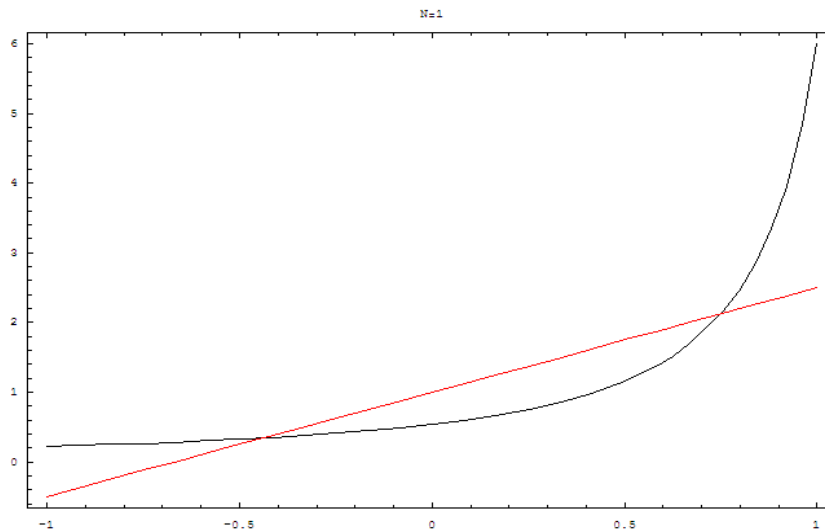
**If N=1** =>

$$\varpi_0^* = \frac{1}{2} \int_{-1}^1 P_{HG}(\cos \Theta) P_0(\cos \Theta) d \cos(\Theta) = \frac{1}{2} \int_{-1}^1 P_{HG}(\cos \Theta) 1 d \cos(\Theta) = \frac{1}{2} \cdot 2 = 1$$

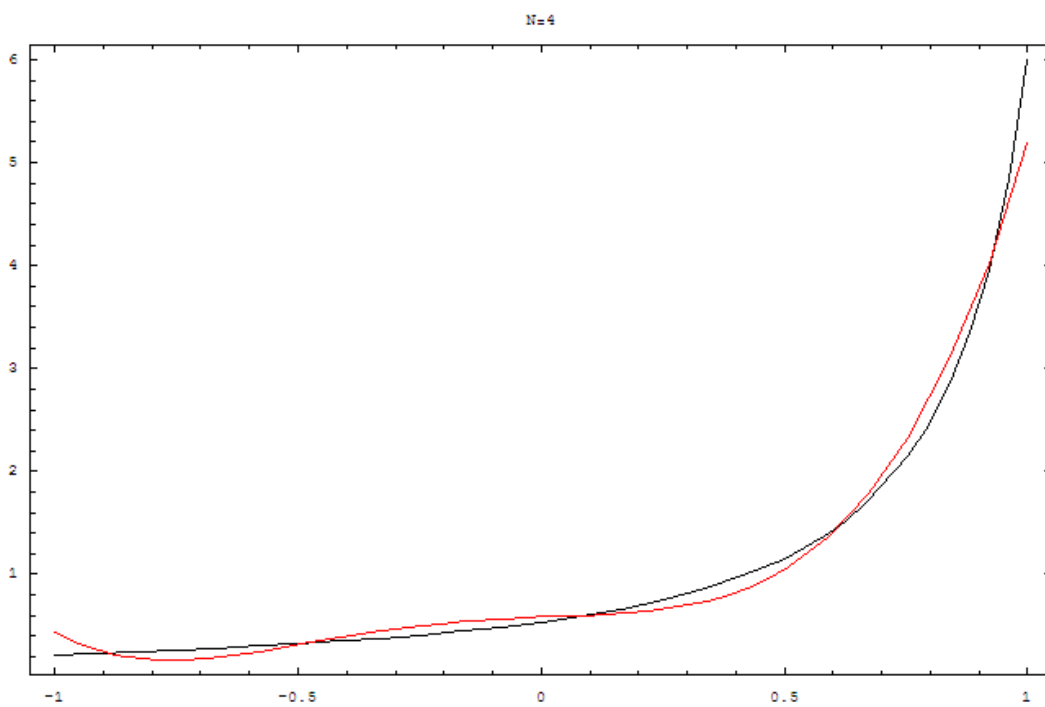
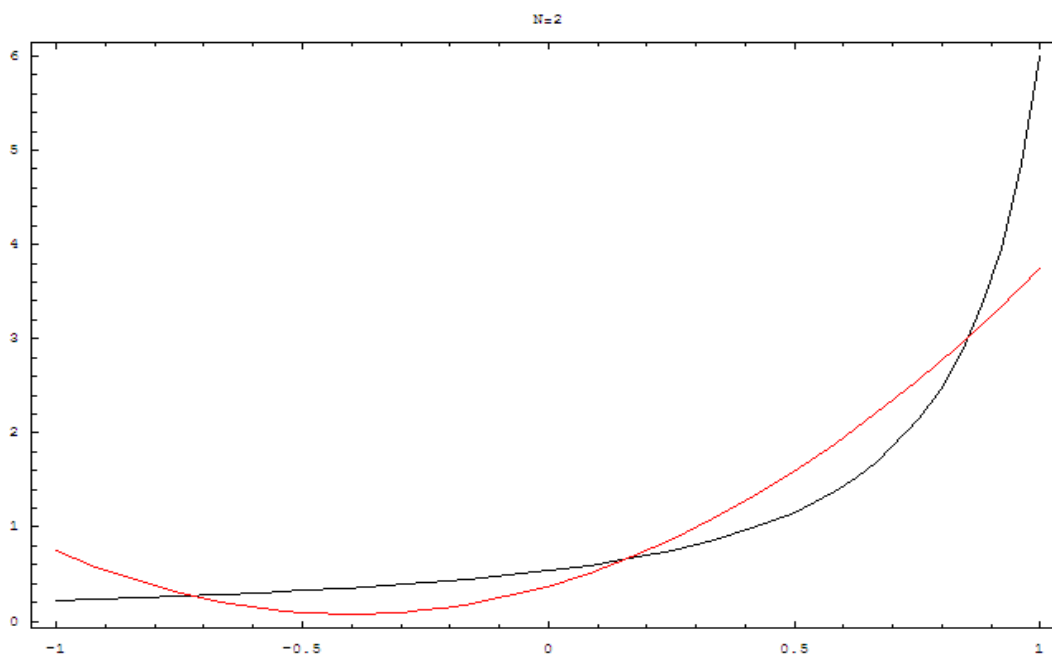
$$\varpi_1^* = \frac{1}{2} \int_{-1}^1 P_{HG}(\cos \Theta) P_1(\cos \Theta) d \cos(\Theta) = \frac{1}{2} \int_{-1}^1 P_{HG}(\cos \Theta) (\cos \Theta) d \cos(\Theta) = 1.5$$

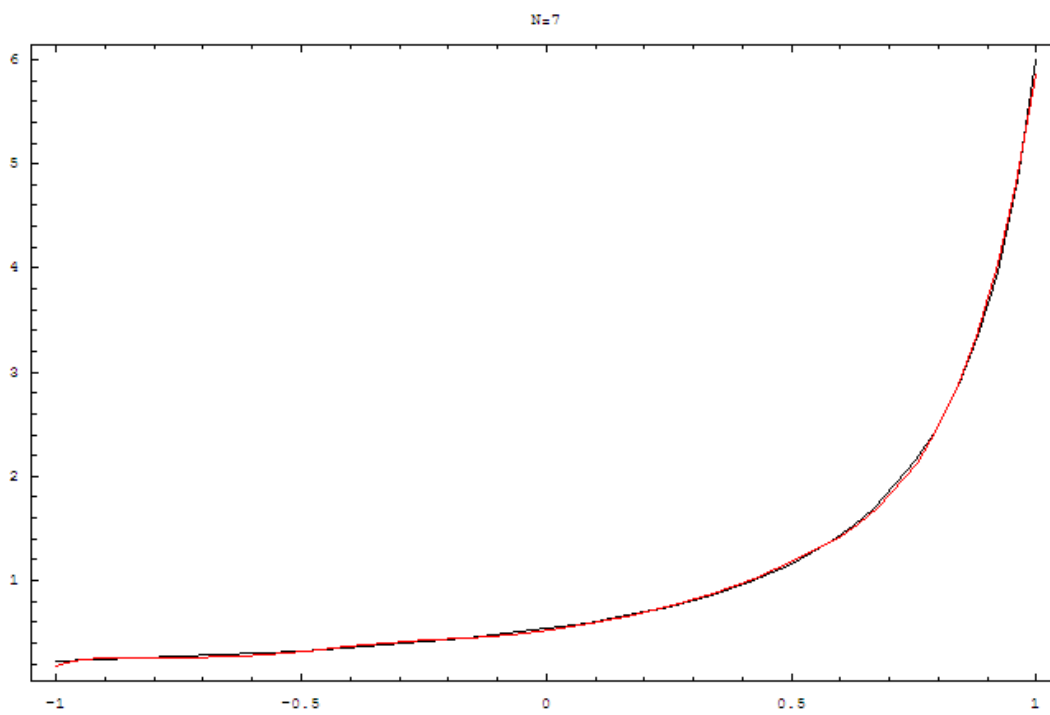
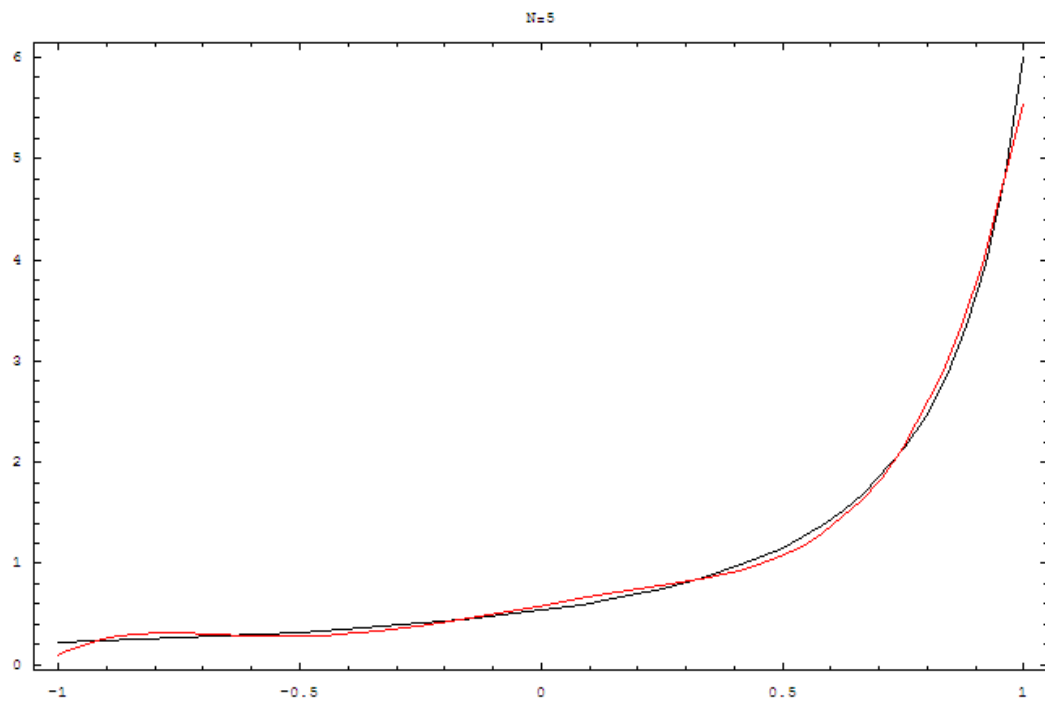
Thus, in the case of two terms in the expansion, we have

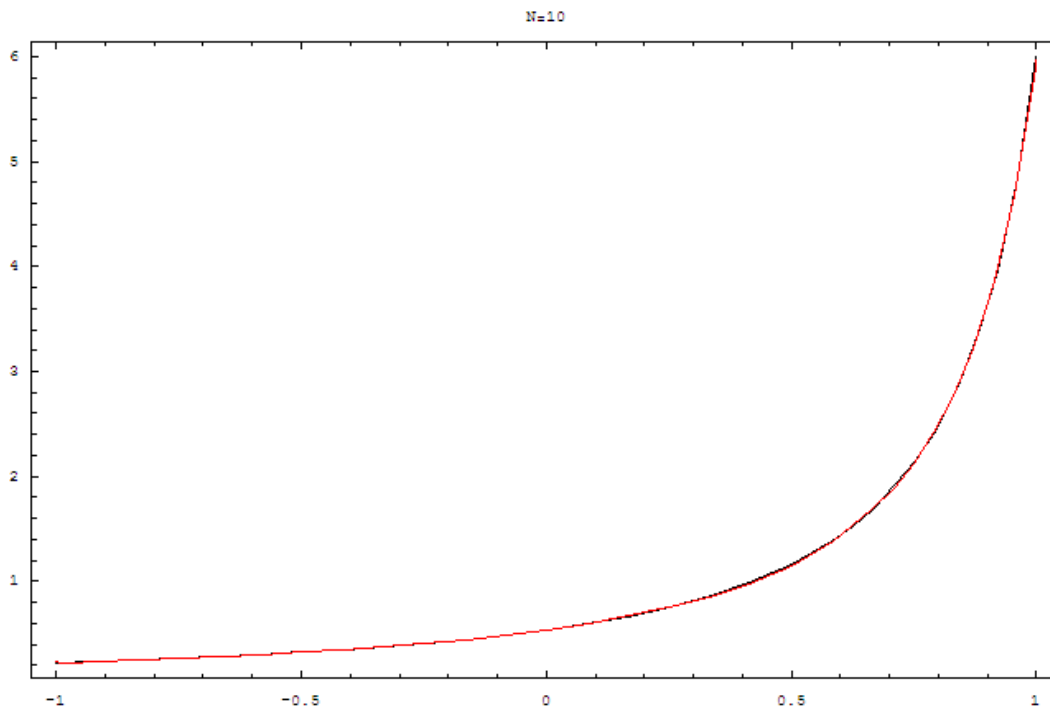
$$P_{HG}(\cos \Theta) \approx \sum_{l=0}^1 \varpi_l^* P_l(\cos \Theta) = 1 + 1.5(\cos \Theta) \quad \text{- still accuracy is not so good!}$$



We can continue by including more terms to get a desirable accuracy







$N=7-10$  gives good approximation to the Henyey-Greenstein scattering phase function with  $g=0.5$ . The larger  $g$ , the larger number of terms will be required to achieve acceptable accuracy.