Blackbody Radiation
Introduction

First radiation process to look at: radiation in thermal equilibrium with itself: blackbody radiation

Assumptions:

1. Photons are Bosons, i.e., more than one photon per phase space cell possible.

2. Photons are in thermodynamic equilibrium at all frequencies.

Outline of computation:

1. Compute mean energy of photons of frequency $\nu$ in phase space cell, $\langle E(\nu) \rangle$

2. Compute number of phase space cells as a function of frequency, $N(\nu)$.

3. Compute photon spectrum as product $\langle E(\nu) \rangle \cdot N(\nu)$. 

Blackbody Radiation: Derivation
First step: Mean energy of photons of frequency \( \nu \) in phase space cell.

Describe phase space cell as box \( \rightarrow \) Photons: \( \sim \) solution of QM harmonic oscillator \( \rightarrow \) Total energy of box with \( n \) photons:

\[
E_n = \left( n + \frac{1}{2} \right) \cdot h\nu
\]  
\( (3.1) \)

where \( \frac{1}{2}h\nu \): ground state energy (unobservable).

Probability that oscillator is in \( n \)th state from Boltzmann:

\[
P_n(\nu, T) = \frac{\exp \left( - \left( n + \frac{1}{2} \right) h\nu \right)}{\sum_{n'} \exp \left( - \left( n' + \frac{1}{2} \right) h\nu \right)} = \frac{\exp(-nh\nu/kT)}{\sum_{n'} \exp(-n'h\nu/kT)}
\]  
\( (3.2) \)

Therefore, average energy per phase cell:

\[
\langle E \rangle = \sum_n E_n P_n(\nu, T)
\]  
\( (3.3) \)

\[
= \sum_n \left\{ \left( n + \frac{1}{2} \right) h\nu \cdot \frac{\exp(-nh\nu/kT)}{\sum_{n'} \exp(-n'h\nu/kT)} \right\}
\]  
\( (3.4) \)

introducing \( x = h\nu/kT \)

\[
= \frac{kT \sum_n \left( n + \frac{1}{2} \right) x \exp(-nx)}{\sum_n \exp(-nx)}
\]  
\( (3.5) \)

\[
= kT \left\{ \frac{\sum_n n x \exp(-nx)}{\sum_n \exp(-nx)} + \frac{x}{2} \right\}
\]  
\( (3.6) \)
Derivation: Step 1, II

To evaluate \( \langle E \rangle \), need to compute the geometric sums \( \sum \exp(-nx) \) and \( \sum n x \exp(-nx) \). We find (see handout)

\[
\sum \exp(-nx) = \frac{1}{1 - \exp(-x)} \quad (3.7)
\]

and

\[
\sum n x \exp(-nx) = \frac{x \exp(-x)}{(1 - \exp(-x))^2} \quad (3.8)
\]

Therefore,

\[
\langle E \rangle = kT \left( \frac{x e^{-x} (1 - e^{-x})^{-2}}{(1 - e^{-x})^{-1}} + \frac{x}{2} \right) \quad (3.9)
\]

\[
= \frac{h\nu \exp(-x)}{1 - \exp(-x)} + \frac{h\nu}{2} \quad (3.10)
\]

\[
= \frac{h\nu}{e^{h\nu/kT} - 1} + \frac{h\nu}{2} \quad (3.11)
\]

We reiterate: the \( h\nu/2 \) term is unobservable \( \Longrightarrow \) Renormalize zero-point of energy to get rid of it.

Could have “known” this result since from Bose-Einstein statistics of particles with chemical potential \( \mu = 0 \) the occupation number is

\[
n_\gamma(\nu, T) = \frac{\langle E \rangle}{h\nu} = \frac{1}{\exp(h\nu/kT) - 1} \quad (3.12)
\]
To prove Eqs. (3.7) and (3.8), look at the Taylor series of \( f(y) = (1 - y)^{-1} \).

By induction:

\[
\begin{align*}
f(y) &= (1 - y)^{-1} \\
\frac{df}{dy} &= \frac{(-1)(-1)}{(1 - y)^2} = \frac{1}{(1 - y)^2} \\
\frac{d^2f}{dy^2} &= \frac{(-1)(-2)}{(1 - y)^3} = \frac{1 \cdot 2}{(1 - y)^3}
\end{align*}
\]

and in general

\[
\frac{d^n f}{dy^n} = \frac{n!}{(1 - y)^{n+1}}
\]  

(3.16)

Therefore, the Taylor series of \( f(y) \) around \( y = 0 \) is

\[
\frac{1}{1 - y} = \sum_n \frac{1}{n!} \frac{d^n f}{dy^n} \bigg|_{y=0} y^n = \sum_n y^n
\]  

(3.17)

Substituting \( y = \exp(-x) \) proves Eq. (3.7).

To prove Eq. (3.8), we need to compute

\[
\sum_n n x \exp(-nx) = x \sum_n n \exp(-nx)
\]  

(3.18)

Note that

\[
\frac{d}{dx} \sum_n \exp(-nx) = - \sum_n n \exp(-nx)
\]  

(3.19)

such that

\[
\sum_n n \exp(-nx) = - \frac{d}{dx} \sum_n \exp(-nx)
\]  

(3.20)

by Eq. (3.7)

\[
= - \frac{d}{dx} \left( \frac{1}{1 - \exp(-x)} \right)
\]  

(3.21)

\[
= \frac{\exp(-x)}{(1 - \exp(-x))^2}
\]  

(3.22)

Multiplying with \( x \) proves Eq. (3.8).
Second Step: Computation of density of phase space cells in box $L_x, L_y, L_z$.

Wave vector of photon:

$$k = \frac{2\pi}{\lambda} n = \frac{2\pi \nu}{c} n$$  

Equation (3.23)

To get all possible photons: count distinguishable photons at same frequency, i.e., photons with different spin or different number of nodes (different $n$).

Spin is easy: there are 2 polarization states

Number of nodes: in the $x, y, z$ direction, number of nodes is

$$n_x = \frac{L_x}{\lambda} = \frac{k_x L_x}{2\pi} \quad \iff \quad dn_x = \frac{L_x}{\lambda} = \frac{L_x \, dk_x}{2\pi}$$  

Equation (3.24)

For $n \gg 1$, can go to “continuum of states”. Eq. 3.24 then implies

$$dN = dn_x \, dn_y \, dn_z = \frac{L_x L_y L_z \, d^3k}{(2\pi)^3} = \frac{V \, d^3k}{(2\pi)^3}$$  

Equation (3.25)

Therefore, the number of states per unit volume per wave number is

$$\frac{n_k}{d^3k} = 2 \cdot \frac{dN}{V \, d^3k} \cdot \frac{1}{(2\pi)^3} = \frac{2}{(2\pi)^3}$$  

Equation (3.26)

Factor 2 from spin.
Because of Eq. (3.23),

\[ d^3k = k^2 \, dk \, d\Omega = \frac{(2\pi)^3}{c^3} \nu^2 \, d\nu \, d\Omega \] (3.27)

such that the 

density of states

\[ \rho_s = \frac{n_{\nu}}{d\nu \, d\Omega} = \frac{2}{(2\pi)^3} \cdot \frac{(2\pi)^3}{c^3} \nu^2 = \frac{2\nu^2}{c^3} \] (3.28)

(number of states per solid angle, per volume, per frequency).
To summarize, we had:

**Mean energy of state:**

\[
\langle E \rangle = \frac{h\nu}{e^{h\nu/kT} - 1}
\]  
(3.11)

**State density:**

\[
\rho_s = \frac{2\nu^2}{c^3}
\]  
(3.28)

The total energy density is then

\[
u_\nu(\Omega) = \langle E \rangle \cdot \rho_s
= \frac{2h\nu^3}{c^3} \frac{1}{\exp(h\nu/kT) - 1}
\]  
(3.29)

(energy per volume per frequency per solid angle)

Because of Eq. (2.30) \((\nu_\nu = I_\nu/c)\), the intensity is given by

\[
I_\nu = \frac{2h\nu^3}{c^2} \frac{1}{\exp(h\nu/kT) - 1} =: B_\nu
\]  
(3.31)

This is the spectrum of a black body.

In \(\lambda\) space, the spectrum is

\[
B_\lambda = \frac{2hc^2/\lambda^5}{\exp(hc/\lambda kT) - 1}
\]  
(3.32)

(since we need \(B_\lambda \, d\lambda = B_\nu \, d\nu\)).
Blackbody Radiation: Properties

The graph shows the spectrum of blackbody radiation for temperatures ranging from $10^3$ K to $10^9$ K. The intensity $I_\nu(T)$ is plotted against frequency $\nu$ in Hz. The intensity is given in units of ergs/s/cm$^2$/Hz/str, with intensity values ranging from $10^{-20}$ to $10^{10}$.
For $h \nu \ll kT \ (\nu \lesssim 2 \times 10^{10} T)$, 

$$\exp \left( \frac{h \nu}{kT} \right) = 1 + \frac{h \nu}{kT} + \ldots \quad (3.33)$$

such that

$$B_\nu \approx \frac{2\nu^2}{c^2} kT \quad (3.34)$$

This is the Rayleigh-Jeans law.

The Rayleigh-Jeans law is used in the radio regime to define the brightness temperature,

$$T_b = I_\nu \cdot \frac{c^2}{2k\nu^2} \quad (3.35)$$

where $I_\nu$ is the measured radio intensity.
For $h\nu \gg kT$, $(\nu \gtrsim 2 \times 10^{10} T)$,

$$\exp \left( \frac{h\nu}{kT} \right) - 1 \sim \exp \left( \frac{h\nu}{kT} \right)$$

(3.36)

such that

$$B_\nu \approx \frac{2h\nu^3}{c^2} \exp \left( -\frac{h\nu}{kT} \right)$$

(3.37)

the Wien spectrum (or Wien’s law).
Wien Displacement Law

The frequency of maximum intensity, $\nu_{\text{max}}$, is obtained by solving

$$\left. \frac{\partial B_{\nu}}{\partial \nu} \right|_{\nu=\nu_{\text{max}}} = 0$$

(3.38)

which is equivalent to solving

$$x = 3 (1 - \exp(-x))$$

(3.39)

where $x = h\nu_{\text{max}}/kT$. Numerically, $x = 2.82$, therefore

$$h\nu_{\text{max}} = 2.82 \cdot kT$$

(3.40)

This is the Wien displacement law.

The frequency of maximum flux is directly proportional to the black body temperature.

Likewise, for $B_{\lambda}$, one finds

$$\lambda_{\text{max}} T = 0.2898 \text{ cm K}$$

(3.41)

Note that $\lambda_{\text{max}}\nu_{\text{max}} \neq c$!

Do not confuse Wien's law and the Wien displacement law...
**Summary: Rayleigh-Jeans vs. Wien**

Rayleigh-Jeans applies for $\nu \lesssim \nu_{\text{max}}$

Wien applies for $\nu \gtrsim \nu_{\text{max}}$.
The total brightness of a black body is obtained from

\[ B(T) = \int_0^\infty B_\nu(T) \, d\nu \]  

(3.42)

... substituting \( x = \frac{h\nu}{kT} \)

\[ = \frac{2h}{c^2} \left( \frac{kT}{h} \right)^4 \int_0^\infty \frac{x^3 \, dx}{\exp(x) - 1} \]  

(3.43)

... the integral has the value \( \pi^4 / 15 \)

\[ = \frac{2\pi^4 k^4}{15c^2 h^3} T^4 = \frac{a c}{4\pi} T^4 = \frac{\sigma_{SB} T^4}{\pi} \]  

(3.44)

Convert the brightness to the flux \( (F = \pi B, \text{ Eq. 2.24}) \), to obtain

\[ F = \sigma_{SB} T^4 \]  

(3.45)

the Stefan-Boltzmann law.

And, yes, Boltzmann’s first name is Ludwig, while Stefan’s first name is Josef.

\( a \) is the radiation density constant,

\[ a := \frac{8\pi^5 k^4}{15c^3 h^3} = 7.566 \times 10^{-15} \text{ erg cm}^{-3} \text{ K}^{-4} \]  

(3.46)

also written as the Stefan-Boltzmann constant

\[ \sigma_{SB} := \frac{2\pi^5 k^4}{15c^2 h^3} = 5.671 \times 10^{-5} \text{ erg cm}^{-2} \text{ K}^{-4} \text{ s} \]  

(3.47)
The effective temperature, $T_{\text{eff}}$, of a spectrum $I_\nu$ is the temperature where

$$F = \int I_\nu \cos \theta \, d\nu \, d\Omega = \sigma T_{\text{eff}}^4$$

(3.48)

Sometimes, $I_\nu$ is only known over a certain wavelength range, and depending on the spectrum the measured $T_{\text{eff}}$ will depend on this range (see figure).
The temperature of an irradiated body is given from energy equilibrium:

\[
\frac{L_\odot}{4\pi a^2} \pi r^2 = \sigma_{SB} T^4 \pi r^2
\]

(3.49)

where \( a \): distance to sun, \( r \): planetary radius.

Therefore

\[
T = \left( \frac{L_\odot}{16\pi \sigma_{SB} r^2} \right)^{1/4} = \frac{281 \text{ K}}{(a/1 \text{ AU})^{1/2}}
\]

(3.50)

Last step used \( L_\odot = 4 \times 10^{33} \text{ erg s}^{-1} \) and \( 1 \text{ AU} = 1.496 \times 10^{13} \text{ cm} \).

If the planet reflects part of the radiation and if the IR emissivity is only roughly a BB, then Eq. (3.49) is modified,

\[
(1 - B)\frac{L_\odot}{4\pi a^2} \pi r^2 = \epsilon \sigma_{SB} T^4 \pi r^2
\]

\[\Rightarrow T = \frac{281 \text{ K}}{(a/1 \text{ AU})^{1/2}} \left( \frac{1 - B}{\epsilon} \right)^{1/4}
\]

(3.51)

where \( B \): Bond albedo, and \( \epsilon \): effective emissivity

For the Earth, \( B = 0.39 \), for Venus, \( B = 0.72 \). Thus, since \( T_{\text{Earth}} \sim 288 \text{ K}, \epsilon_{\text{Earth}} = 0.55 < 1 \) (greenhouse effect).

If the planet is not a fast rotator, replace \( 4\pi r^2 \) by \( 2\pi r^2 \).